Math 261 Practice Exam (9/25/09)
Ch. 13 and 14.1, 14.3-14.5

1. Let \( \mathbf{r}(t) = \langle \sqrt{2-t}, \frac{e^t-1}{t}, \ln(t+1) \rangle \). Find:
   a) The domain of \( \mathbf{r}(t) \).
   b) The limit of \( \mathbf{r}(t) \) as \( t \to 0 \).
   c) The derivative \( \mathbf{r}'(t) \).

   Ans: a) \(-1 < t < 0 \) or \( 0 < t \leq 2 \).
   b) \( \langle \sqrt{2}, 1, 0 \rangle \)
   c) \( \langle \frac{1}{t^2(\sqrt{2-t})}, \frac{1}{t+1} \rangle \)

2. Find parametric equations for the tangent line to the curve \( x = 2 \sin(t), y = 2 \sin(2t), z = 2 \sin(3t) \) at the point \((1, \sqrt{3}, 2)\).

   Ans. \( x = 1+\sqrt{3}t, y = \sqrt{3} + 2t, z = 2 \).

3. Give the general formula for the length of a space curve \( \mathbf{r}(t) \), and use it to find the length of the curve \( \mathbf{r}(t) = \langle 2t^{3/2}, \cos(2t), \sin(2t) \rangle \) for \( 0 \leq t \leq 1 \).

   Ans. For original exponent on \( t \) (1/2), answer is \( \sqrt{5} + \frac{1}{4} \ln (9 + 4\sqrt{5}) \) (use Maple).
   For corrected exponent on \( t \) (namely 3/2), the answer is \( \frac{2}{27} (13^{3/2} - 8) \).

4. The helix \( \mathbf{r}_1(t) = \cos(t) \mathbf{i} + \sin(t) \mathbf{j} + t \mathbf{k} \) intersects the curve \( \mathbf{r}_2(t) = (1 + t) \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k} \) at the point \((1, 0, 0)\). Find the angle of intersection of these curves. How is this angle defined?

   Ans. \( \pi/2 \)

5. Give formulas for the tangential and normal components of acceleration along some trajectory \( \mathbf{r}(t) \). Find the tangential and normal components of acceleration of the trajectory \( \mathbf{r}(t) = \cos(t) \mathbf{i} + \sin(t) \mathbf{j} + t \mathbf{k} \) at time \( t \).

   Ans. Tangential = \( aT = 0 \). Normal = \( aN = 1 \), so acceleration vector \( = N \).

6. Give the general formula for curvature and use it to find the curvature at time \( t \) of the trajectory in problem 5.

   Ans. Curvature = \( 1/2 \) for all \( t \).
7. Find the parametric equations of the osculating circle of the curve in problem 5 at time $t = \pi/2$.

Ans. $x = -\sqrt{2}\cos (u)$, $y = -2\sin(u) - 1$, $z = \sqrt{2} \cos(u) + \frac{\pi}{2}$.

8. Find the equation of the osculating plane of the curve in problem 5 at time $t = \pi/2$.

Ans. $\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}(z - \frac{\pi}{2}) = 0$.

9. Find the partial derivatives $f_x, f_y, f_{xx}, f_{yy}$ and $f_{xy}$ for the function $f(x, y) = \ln(e^x + e^y)$.

Verify that $z = f(x, y)$ is a solution of the equations $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$ and

$$\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \frac{e^x e^y}{(e^x + e^y)^2} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y}\right) - \frac{e^{x+y}}{(e^x + e^y)^2}$$.

Ans. $\frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^y}$, $\frac{\partial z}{\partial y} = \frac{e^y}{e^x + e^y}$, $\frac{\partial^2 z}{\partial x^2} = \frac{e^x e^y}{(e^x + e^y)^2} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y}\right)$, $\frac{\partial^2 z}{\partial y^2} = \frac{e^x e^y}{(e^x + e^y)^2} \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x}\right)$, etc., etc.

10. Find the equation of the tangent plane to the surface $z = x^2 + 6y^2$ at the point $(2, 1, \sqrt{10})$.

Ans. $z = 4x + 12y - 20 + \sqrt{10}$.

11. Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at $(7, 2)$ and use it to approximate $f(6.9, 2.06)$.

Ans. $L(x, y) = 1(x - 7) - 3(y - 2) + 0$. $L(6.9, 1.06) = -0.28$.

12. Give the definition of differentiability for the function $w = f(x, y, z)$ at a point $(a, b, c)$.

Ans. $\Delta w = f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z + \epsilon_1\Delta x + \epsilon_2\Delta y + \epsilon_3\Delta z$, etc., etc.

13. Use the chain rule to find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = \arcsin(x - y)$, $x = s^2 + t^2$, $y = 1 - 2st$.

Ans. $\partial z/\partial s = \frac{2s + 2t}{R} = \partial z/\partial t$. (Use symmetry in $s$ and $t$.)

14. In the first step of the proof of Kepler’s first law of planetary motion, explain why the vector $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ must be constant. Ans. See p. 881, top of the page.